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# Numerical radius: New Extensions and Inequalities

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**Abstract**

We firstly define a seminorm on the space of bounded linear operators on a Hilbert space, which generalizes the numerical radius norm. We investigate basic properties of this seminorm and prove inequalities involving it. Further, for a positive element $a$ in a unital $C^*$-algebra $\mathfrak{A}$ we define a semi-norm on $\mathfrak{A}$, which generalizes the $a$-operator semi-norm and the $a$-numerical radius.

**Keywords:** Numerical range, numerical radius, inequality.

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## 1 Introduction and preliminaries

Let $\mathfrak{A}$ be a $C^*$-algebra with unit denoted by $\mathbf{1}$ and let $a \in \mathfrak{A}$ be a positive element. Let $\mathbb{B}(\mathcal{H})$ be the $C^*$-algebra of all bounded linear operators on a complex Hilbert space $\big(\mathcal{H}, \langle\cdot,\cdot\rangle\big)$. By a state on $\mathfrak{A}$ we mean a positive linear functional $f$ on $\mathfrak{A}$ such that $\|f\| = 1$ and let $\mathcal{S}(\mathfrak{A})$ denote the set of states on $\mathfrak{A}$. For an element $x \in \mathfrak{A}$, let $V(x)$ denote the (algebraic) numerical range of $x \in \mathfrak{A}$, that is, the set $V(x) = \big\{f(x) : f \in \mathcal{S}(\mathfrak{A})\big\}$. This set generalizes the classical numerical range in the sense that the numerical range $V(T)$ of a Hilbert space operator $T$ (considered as an element of a $C^*$-algebra $\mathbb{B}(\mathcal{H})$) coincides with the closure of its classical numerical range $W(T) = \big\{\langle T\xi, \xi\rangle : \xi \in \mathcal{H}, \|\xi\| = 1\big\}$. The numerical radius of $x \in \mathfrak{A}$ is defined as $v(x) = \sup\big\{|\lambda| : \lambda \in V(x)\big\}$.

Recently, Bourhim and Mabrouk in [3] introduced and studied $a$-numerical range and $a$-numerical radius of elements in $C^*$-algebras. Also, the authors in [1] continued the work on the $a$-numerical range and the $a$-numerical radius. In particular, some ideas from the recent papers are extended.

Set $\mathcal{S}_a(\mathfrak{A}) = \Big\{\frac{f}{f(a)} : f \in \mathcal{S}(\mathfrak{A}),\, f(a) \neq 0\Big\}$ and for an element $x \in \mathfrak{A}$, let $\|x\|_a = \sup\Big\{\sqrt{f(x^*ax)} : f \in \mathcal{S}_a(\mathfrak{A})\Big\}$. It is worth observing that $\|\cdot\|_{\mathbf{1}} = \|\cdot\|$ and $\|x\|_a = 0$ if and only if $ax = 0$. Further the set $\mathcal{S}_a(\mathfrak{A})$ is a non empty, convex and closed subset of the topological dual space of $\mathfrak{A}$, but it is compact if and only if $a$ is invertible in $\mathfrak{A}$; see [3, proposition 2.3]. In particular if $a$ is not invertible and due to the lack of compactness of $\mathcal{S}_a(\mathfrak{A})$, it may happen that $\|x\|_a = \infty$ for some $x \in \mathfrak{A}$; see [3, Example 3.2]. In the sequel we will denote $\mathfrak{A}^a = \{x \in \mathfrak{A} : \|x\|_a < \infty\}$. The set $\mathfrak{A}^a$ is a subalgebra of $\mathfrak{A}$ not necessarily closed. Also by [3, Proposition 3.3], $\|\cdot\|_a$ is a semi-norm on $\mathfrak{A}^a$ and satisfies

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$\|xy\|_a \leq \|x\|_a \|y\|_a$ for all $x, y \in \mathfrak{A}^a$. Denote by $\mathfrak{A}_a$ the set of all elements in $\mathfrak{A}$ that admit $a$-adjoints. Recall that for an element $x \in \mathfrak{A}$, an element $x^{\sharp_a} \in \mathfrak{A}$ is said to be an $a$-adjoint of $x$ if $ax^{\sharp_a} = x^*a$. Basic properties of $\mathfrak{A}_a$ were investigated in [3]. In particular, $\mathfrak{A}_a$ is a subalgebra of $\mathfrak{A}^a$ which is neither closed nor dense in $\mathfrak{A}$. Further if $x \in \mathfrak{A}_a$ and $x^{\sharp_a}$ is an $a$-adjoint of it, then by

$$\|x\|_a^2 = \left\|xx^{\sharp_a}\right\|_a = \left\|x^{\sharp_a}x\right\|_a = \left\|x^{\sharp_a}\right\|_a^2. \tag{1}$$

An element $x \in \mathfrak{A}$ is said to be $a$-self-adjoint if $ax$ is self-adjoint, i.e., $ax = x^*a$. We say that $x$ is $a$-positive if $ax$ is positive. Every element $x$ in $\mathfrak{A}_a$ can be written as $x = y + iz$ where $y$ and $z$ are $a$-self-adjoint but, in general, this decomposition is not unique. In fact if $x^{\sharp_a}$ is an $a$-adjoint of $x$, then $x = \mathfrak{R}(x) + i\mathfrak{I}(x)$, where $\mathfrak{R}(x) = \frac{x + x^{\sharp_a}}{2}$ and $\mathfrak{I}(x) = \frac{x - x^{\sharp_a}}{2i}$ are $a$-real and $a$-imaginary parts of $x$, respectively. The $a$-numerical range (respectively, the $a$-numerical radius) of an element $x \in \mathfrak{A}$ are defined by $V_a(x) = \{f(ax) : f \in \mathcal{S}_a(\mathfrak{A})\}$ (respectively, $v_a(x) = \sup\{|\lambda| : \lambda \in V_a(x)\}$). In contrast of the classical algebraic numerical range, the $a$-numerical range $V_a(x)$ of $x \in \mathfrak{A}$ may be unbounded. Note that these concepts were introduced in [3] as generalizations of the $A$-numerical range (respectively, the $A$-numerical radius) for Hilbert space operator $T$ given by $W_A(T) = \{\langle AT\xi, \xi\rangle : \xi \in \mathcal{H}, \|\xi\|_A = 1\}$ (respectively, $w_A(T) = \sup\{|\lambda| : \lambda \in W_A(T)\}$), where $A$ is a positive operator on $\mathcal{H}$ and $\|\xi\|_A = \sqrt{\langle A\xi, \xi\rangle}$ for all $\xi \in \mathcal{H}$. In particular, when $A$ is the identity operator on $\mathcal{H}$, then $A$-numerical range and $A$-numerical radius of $T$ coincide with the classical numerical range and numerical radius, respectively, i.e., $W_A(T) = W(T)$ and $w_A(T) = w(T)$.

An important and useful identity for the $a$-numerical radius (see [3, Theorem 4.11]) is as follows:

$$v_a(x) = \sup_{\theta \in \mathbb{R}} \left\|\mathfrak{R}(e^{i\theta}x)\right\|_a.$$

By [3, Proposition 3.3 and Corollary 4.10], observe that $v_a(\cdot)$ defines a semi-norm on $\mathfrak{A}_a$, which is equivalent to the $a$-operator semi-norm $\|\cdot\|_a$. Namely, for $x \in \mathfrak{A}_a$, it holds that

$$\frac{1}{2}\|x\|_a \leq v_a(x) \leq \|x\|_a. \tag{2}$$

The first inequality becomes equality if $ax \neq 0$ and $ax^2 = 0$ and the second inequality becomes equality if $x$ is $a$-self-adjoint (see, [3, Corollary 4.6]).

## 2 A generalization of the numerical radius for Hilbert space operators

The notion of orthogonality in an arbitrary normed linear space may be introduced in various ways. Among them, the one which is frequently studied in literature is the *Birkhoff–James orthogonality* [2,4]: if $x, y$ are elements of a normed linear space $E$ equipped with the norm $N(\cdot)$, then $x$ is orthogonal to $y$ in the Birkhoff–James sense, in short $x \perp_B^N y$, if

$$N(x + \lambda y) \geq N(x), \quad \forall \lambda \in \mathbb{C}.$$

Moreover, $\|\cdot\|_N^* \colon E^* \longrightarrow [0, +\infty)$ stands for the dual norm, i.e. $\|\cdot\|_N^*$ is a norm in $E^* = \big(E, N(\cdot)\big)^*$. For fixed $x \in E$ let $J_N(x)$ denote the set of its supporting functionals:

$$J_N(x) := \Big\{f \in E^* : \|f\|_N^* = 1,\, f(x) = N(x)\Big\}.$$

The Hahn-Banach theorem implies that $J_N(x) \neq \emptyset$. Recall that a unit vector point $u \in E$ is called a *vertex* of the closed unit ball in $E$ if $J_N(u)$ is total over $E$.

Now, let $N(\cdot)$ be an arbitrary norm on $\mathbb{B}(\mathcal{H})$. According to the beginning of this section, for fixed $T \in \mathbb{B}(\mathcal{H})$ we have

$$J_N(T) = \Big\{f \in \mathbb{B}(\mathcal{H})^* : \|f\|_N^* = 1, \, f(T) = N(T)\Big\}.$$

Since Birkhoff-James orthogonality has the property of right existence, we obtain $\big\{\xi \in \mathbb{C} : I \perp_B^N (T - \xi I)\big\} \neq \emptyset$. Let $I \perp_B^N (T - \xi I)$ for some $\xi \in \mathbb{C} \setminus \{0\}$. Hence $N\big(I + \frac{1}{\xi}(T - \xi I)\big) \geq N(I)$ and so $|\xi| \leq \frac{N(T)}{N(I)}$. Thus the set $\big\{\xi \in \mathbb{C} : I \perp_B^N (T - \xi I)\big\}$ is also bounded in $\mathbb{C}$. This motivates the following definition (see [9]).

**Definition 2.1.** Let $N(\cdot)$ be a norm on $\mathbb{B}(\mathcal{H})$. The function $w_N : \mathbb{B}(\mathcal{H}) \to [0, +\infty)$ is defined as

$$w_N(T) := \sup\Big\{|\xi| : \xi \in \mathbb{C}, \, I \perp_B^N (T - \xi I)\Big\}$$

for every $T \in \mathbb{B}(\mathcal{H})$.

**Remark 2.2.** Let $N(\cdot)$ be a norm on $\mathbb{B}(\mathcal{H})$ and let $T \in \mathbb{B}(\mathcal{H})$. For every $\xi \in \mathbb{C}$, we have

$$\begin{aligned} I \perp_B^N (T - \xi I) &\iff N\big(I + \lambda(T - \xi I)\big) \geq N(I) \quad \forall \lambda \in \mathbb{C} \\ &\iff N\big(I + \frac{1}{\xi - \lambda}(T - \xi I)\big) \geq N(I) \quad \forall \lambda \in \mathbb{C} \setminus \{\xi\} \\ &\iff N\big((\xi - \lambda)I + T - \xi I\big) \geq |\xi - \lambda| N(I) \quad \forall \lambda \in \mathbb{C} \\ &\iff N\big(T - \lambda I\big) \geq |\xi - \lambda| N(I) \quad \forall \lambda \in \mathbb{C}. \end{aligned}$$

Thus

$$I \perp_B^N (T - \xi I) \iff N\big(T - \lambda I\big) \geq |\xi - \lambda| N(I) \quad \forall \lambda \in \mathbb{C}. \tag{3}$$

**Remark 2.3.** For any $T \in \mathbb{B}(\mathcal{H})$, it is well-known (see [7]) that

$$\overline{W(T)} = \bigcap_{\lambda \in \mathbb{C}} \big\{\xi : \|T - \lambda I\| \geq |\xi - \lambda|\big\}.$$

Therefore, by (3), we have

$$w(T) = \sup\Big\{|\xi| : \xi \in \mathbb{C}, \, I \perp_B^{\|\cdot\|} (T - \xi I)\Big\}.$$

In view of the previous relation, it is now obvious that $w_N(\cdot)$ generalizes the classical numerical radius $w(\cdot)$.

**Proposition 2.4.** *Let $N(\cdot)$ be a norm on $\mathbb{B}(\mathcal{H})$ and let $T \in \mathbb{B}(\mathcal{H})$. Then the following properties hold:*

*(i)* $w_N(I) = 1$.

*(ii)* $w_N(T) \leq \frac{N(T)}{N(I)}$.

*(iii) If $N(\cdot)$ is self-adjoint, then so is $w_N(\cdot)$.*

*(iv) If $N(\cdot)$ is weakly unitarily invariant, then so is $w_N(\cdot)$.*

**Theorem 2.5.** *Let $N(\cdot)$ be a norm on $\mathbb{B}(\mathcal{H})$. Then $w_N(\cdot)$ is a seminorm on $\mathbb{B}(\mathcal{H})$.*

**Remark 2.6.** Let $N(\cdot)$ be an arbitrary norm on $\mathbb{B}(\mathcal{H})$. By Theorem 2.5, $w_N(\cdot)$ is a seminorm on $\mathbb{B}(\mathcal{H})$. Therefore, for $T \in \mathbb{B}(\mathcal{H})$, if $T = 0$, then $w_N(T) = 0$. The converse is however not true, in general (see Theorem 2.7).

From now on we assume that the considered norm $N\colon \mathbb{B}(\mathcal{H}) \longrightarrow [0, +\infty)$ satisfies $N(I) = 1$. There is no loss in generality in assuming this. In particular, the classical norms on $\mathbb{B}(\mathcal{H})$ satisfy such equality for the identity operator $I$. Therefore, we think that such assumption is interesting for investigations.

Now, we are going to prove a condition for checking when $w_N(\cdot)$ is a norm on $\mathbb{B}(\mathcal{H})$.

**Theorem 2.7.** *Let $N(\cdot)$ be a norm on $\mathbb{B}(\mathcal{H})$ with $N(I) = 1$. The following conditions are equivalent:*

*(i) $w_N(\cdot)$ is a norm on $\mathbb{B}(\mathcal{H})$.*

*(ii) The operator $I$ is a vertex of the closed unit ball in $\big(\mathbb{B}(\mathcal{H}), N(\cdot)\big)$.*

The following result says that the spaces $\big(\mathbb{B}(\mathcal{H}), N(\cdot)\big)$ and $\big(\mathbb{B}(\mathcal{H}), w_N(\cdot)\big)$ are similar (in some sense) in the point $I$.

**Theorem 2.8.** *Let $N(\cdot)$ be a norm on $\mathbb{B}(\mathcal{H})$ with $N(I) = 1$. If $w_N(\cdot)$ is a norm on $\mathbb{B}(\mathcal{H})$, then*

$$J_N(I) = J_{w_N}(I). \tag{4}$$

*In this case, the operator $I$ is a vertex of the closed unit ball in $\big(\mathbb{B}(\mathcal{H}), w_N(\cdot)\big)$.*

Now we may consider the function $w_{w_N}\colon \mathbb{B}(\mathcal{H}) \longrightarrow [0, +\infty)$. Suppose that $w_N(\cdot)$ is a norm on $\mathbb{B}(\mathcal{H})$. It follows from Proposition 2.4(i)-(ii) that $w_{w_N}(\cdot) \leq w_N(\cdot)$. Moreover, Theorem 2.5 yields the subadditivity of $w_{w_N}(\cdot)$. It is amazing that these remarks can be strengthen as follows.

**Theorem 2.9.** *Let $N(\cdot)$ be a norm on $\mathbb{B}(\mathcal{H})$ with $N(I) = 1$. If $w_N(\cdot)$ is a norm on $\mathbb{B}(\mathcal{H})$, then $w_{w_N}(\cdot)$ is also a norm on $\mathbb{B}(\mathcal{H})$. Moreover, $w_{w_N}(\cdot) = w_N(\cdot)$.*

Our next result reads as follows.

**Theorem 2.10.** *Let $N(\cdot)$ is a weakly unitarily invariant norm on $\mathbb{B}(\mathcal{H})$ and let $T$ and $S$ be self-adjoint operator in the norm-unit ball of $\mathbb{B}(\mathcal{H})$. Then*

$$w_N(TS \pm ST) \leq \sup_{U \in \mathcal{U}} \big\{ w_N(TU \pm U^*T), w_N(SU \pm U^*S) \big\},$$

*where $\mathcal{U}$ is the unitary group of all unitary operators in $\mathbb{B}(\mathcal{H})$.*

As a consequence of Theorem 2.10, we have the following result.

**Corollary 2.11.** *Let $N(\cdot)$ is a weakly unitarily invariant norm on $\mathbb{B}(\mathcal{H})$ and let $T$ be an operator in the norm-unit ball of $\mathbb{B}(\mathcal{H})$. Then*

$$w_N(TT^* - T^*T) \leq 2 \sup_{U \in \mathcal{U}} \Big\{ w_N\big(\mathfrak{R}(T)U \pm U^*\mathfrak{R}(T)\big), w_N\big(\mathfrak{I}(T)U \pm U^*\mathfrak{I}(T)\big) \Big\},$$

*where $\mathcal{U}$ is the unitary group of all unitary operators in $\mathbb{B}(\mathcal{H})$.*

## 3 An extension of the $a$-numerical radius on $C^*$-algebras

First, let us define notions weighted $a$-real and $a$-imaginary parts of elements in $\mathfrak{A}_a$. Let $s$ and $t$ be two nonnegative reals such that $s+t>0$. Define the weighted $a$-real and $a$-imaginary parts of $x\in\mathfrak{A}_a$ by $\mathfrak{R}_{(s,t)}(x)=sx+tx^{\sharp_a}$ and $\mathfrak{I}_{(s,t)}(x)=s(-ix)+t(-ix)^{\sharp_a}$, respectively. When $s=t=\frac{1}{2}$, we clearly have $\mathfrak{R}_{(\frac{1}{2},\frac{1}{2})}(x)=\mathfrak{R}(x)$ and $\mathfrak{I}_{(\frac{1}{2},\frac{1}{2})}(x)=\mathfrak{I}(x)$. Also define the function $v_{(a,(s,t))}(\cdot)\colon\mathfrak{A}_a\longrightarrow[0,+\infty)$ by

$$v_{(a,(s,t))}(x)=\sup_{\theta\in\mathbb{R}}\left\|\mathfrak{R}_{(s,t)}(e^{i\theta}x)\right\|_a. \tag{5}$$

**Remark 3.1.** For $x\in\mathfrak{A}_a$, it is easy to see that $v_{(a,(s,t))}(x)=\sup_{\theta\in\mathbb{R}}\left\|\mathfrak{I}_{(s,t)}(e^{i\theta}x)\right\|_a$.

**Remark 3.2.** Obviously, $v_{(a,(1,0))}(x)=v_{(a,(0,1))}(x)=\|x\|_a$, and $v_{(a,(\frac{1}{2},\frac{1}{2}))}(x)=v_a(x)$. Hence $v_{(a,(s,t))}(\cdot)$ generalizes the $a$-operator semi-norm $\|\cdot\|_a$ and the $a$-numerical radius $v_a(\cdot)$, which have been introduced in [3].

**Remark 3.3.** Let $\mathfrak{A}=\mathbb{B}(\mathcal{H})$ and let $0\le\nu\le1$. We have

$$v_{(I,(\nu,1-\nu))}(T)=\sup_{\theta\in\mathbb{R}}\left\|\nu e^{i\theta}T+(1-\nu)(e^{i\theta}T)^*\right\|:=w_\nu(T).$$

Thus $v_{(a,(s,t))}(\cdot)$ also generalizes the weighted numerical radius $w_\nu(\cdot)$, which has been recently introduced in [6] (see also [8]).

Our first result reads as follows.

**Theorem 3.4.** *Let $x\in\mathfrak{A}_a$. The following statements hold.*

*(i)* $v_{(a,(s,t))}(x)=\sup_{\alpha,\beta\in\mathbb{R},\alpha^2+\beta^2=1}\left\|\alpha\,\mathfrak{R}_{(s,t)}(x)+\beta\,\mathfrak{I}_{(s,t)}(x)\right\|_a$.

*(ii)* $v_{(a,(s,t))}(x)=\frac{1}{2}\sup_{\theta,\varphi\in\mathbb{R}}\left\|\mathfrak{R}_{(s,t)}\left((e^{i\theta}-ie^{i\varphi})x\right)\right\|_a$.

The next result establishes that $v_{(a,(s,t))}(\cdot)$ and $\|\cdot\|_a$ are two equivalent semi-norm on $\mathfrak{A}_a$.

**Theorem 3.5.** *$v_{(a,(s,t))}(\cdot)$ is a semi-norm on $\mathfrak{A}_a$ and for every $x\in\mathfrak{A}_a$ the following inequalities hold:*

$$\max\{s,t\}\|x\|_a\le v_{(a,(s,t))}(x)\le(s+t)\|x\|_a. \tag{6}$$

**Remark 3.6.** For $x\in\mathfrak{A}_a$, by (1), we have

$$\begin{aligned}
v_{(a,(s,t))}(x^{\sharp_a})&=\sup_{\theta\in\mathbb{R}}\left\|se^{i\theta}x^{\sharp_a}+te^{-i\theta}(x^{\sharp_a})^{\sharp_a}\right\|_a\\
&=\sup_{\theta\in\mathbb{R}}\left\|\left(se^{-i\theta}x+te^{i\theta}x^{\sharp_a}\right)^{\sharp_a}\right\|_a\\
&=\sup_{\theta\in\mathbb{R}}\left\|se^{-i\theta}x+te^{i\theta}x^{\sharp_a}\right\|_a=v_{(a,(s,t))}(x),
\end{aligned}$$

and hence $v_{(a,(s,t))}(x^{\sharp_a})=v_{(a,(s,t))}(x)$.

In the following result, we give a condition equivalent to $v_{(a,(s,t))}(x) = \max\{s,t\}\|x\|_a$.

**Theorem 3.7.** *Let $x \in \mathfrak{A}_a$. The following are equivalent:*

*(i)* $\left\|\mathfrak{R}_{(s,t)}(e^{i\theta}x)\right\|_a = \max\{s,t\}\|x\|_a$ *for all* $\theta \in \mathbb{R}$.

*(ii)* $v_{(a,(s,t))}(x) = \max\{s,t\}\|x\|_a$.

In the following theorem, a refinement of the inequality (6) is given.

**Theorem 3.8.** *Let $x \in \mathfrak{A}_a$. Then*

$$v_{(a,(s,t))}(x) \leq \sqrt{(s^2+t^2)\|x\|_a^2 + 2st\, v_a(x^2)} \leq (s+t)\|x\|_a.$$

**Corollary 3.9.** *If $x \in \mathfrak{A}_a$ is such that $v_{(a,(s,t))}(x) = (s+t)\|x\|_a$, then $\|x^2\|_a = \|x\|_a^2$.*

Our final result extends and refines an inequality for the numerical radius of Hilbert space operators obtained by Kittaneh in [5].

**Theorem 3.10.** *Let $x \in \mathfrak{A}_a$. Then*

$$st\left\|xx^{\sharp_a} + x^{\sharp_a}x\right\|_a + \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left|\left\|\mathfrak{R}_{(s,t)}(e^{i\theta}x)\right\|_a^2 - \left\|\mathfrak{I}_{(s,t)}(e^{i\theta}x)\right\|_a^2\right| \leq v^2_{(a,(s,t))}(x).$$

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